14. Quasi-coherent sheaves

Last chapter

X scheme $\rightsquigarrow$ $\mathcal{F}$ sheaf of modules on X (+ constructions)

Problems

↳ arbitrary $\mathcal{F}$: too general to work with
↳ where to get examples?

Idea

R ring $\rightsquigarrow$ $X = \operatorname{Spec} R$ scheme
M an R-module $\rightsquigarrow$ $\tilde{M}$ an $\mathcal{O}_{\operatorname{Spec} R}$-module

↳ will restrict to $\mathcal{O}_X$-modules locally of this form

quasi-coherent sheaves

Def (Sheaf associated to a module)
R ring, M an R-module, $U \subseteq \operatorname{Spec} R$ open.
Define the set $\tilde{M}(U)$ as collections

$$\varphi = (\varphi_p)_{p \in U} \quad \text{with } \varphi_p \in M_p \text{ for all } p \in U$$

such that:

$$\left.\begin{array}{l} \forall p \in U \; \exists f \in R, g \in M \text{ and } p \in \underset{\text{open}}{U_p} \subseteq U: \\ f \notin q \text{ and } \varphi_q = \dfrac{g}{f} \in M_q \quad \forall q \in U_p. \end{array}\right\} (*)$$

$\rightsquigarrow$ $M = R$ $\rightsquigarrow$ recover definition of $\mathcal{O}_{\operatorname{Spec} R}(U) \Rightarrow \tilde{R} = \mathcal{O}_{\operatorname{Spec} R}$
$\rightsquigarrow$ $\tilde{M}$ sheaf, $\tilde{M}(U)$ is $\tilde{R}(U)$-module $\Rightarrow$ $\tilde{M}$ is $\mathcal{O}_{\operatorname{Spec} R}$-module

sheaf associated to M

Pro $X = \operatorname{Spec} R$ affine scheme, M an R-module

(a) $\forall p \in X$: $\widetilde{M}_p \cong M_p$

stalk of $\widetilde{M}$ at p — localization of M at prime ideal $p \trianglelefteq R$

(b) $\forall f \in R$: $\widetilde{M}(D(f)) \cong M_f$. $\xRightarrow{f=1}$ $\widetilde{M}(X) = M$.

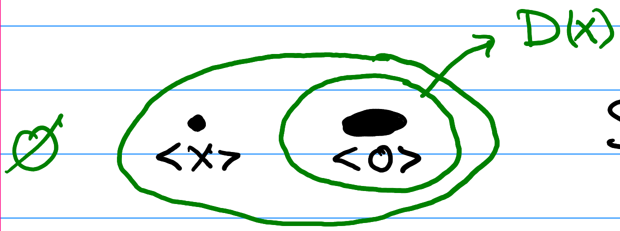
Pf $M = R \rightsquigarrow$ proven in [Lem. 12.18, Pro 12.19]

M arbitrary $\rightsquigarrow$ repeat same proofs

(have not multiplied numerators of fractions) $\square$

Not every $\mathcal{O}_{\operatorname{Spec} R}$-module is of form $\widetilde{M}$:

Exa $R = K[x]_{\langle x \rangle} \cong \mathcal{O}_{\mathbb{A}^1, 0} \rightsquigarrow X = \operatorname{Spec} R$ describes $\mathbb{A}^1$ at origin



Spec R

open sets in Spec R

For $U = D(x)$: define presheaf of modules $\mathcal{F}$ on X:

$$\begin{array}{ll}
\mathcal{F}(X) = \{0\} & R\text{-module} \\
\downarrow \rho_{X,U} & \\
\mathcal{F}(U) = K[x]_{\langle 0 \rangle} & R_x = K[x]_{\langle 0 \rangle}\text{-module} \\
\downarrow \rho_{U,\emptyset} & \\
\mathcal{F}(\emptyset) = \{0\} & 0\text{-module}
\end{array}$$

$\rightsquigarrow \mathcal{F}$ is sheaf ($\nexists$ nontrivial open covers)

But $\mathcal{F} \neq \widetilde{M}$ since otherwise $M = \mathcal{F}(X) = \{0\}$ $\lightning$

Quasi-coherent sheaves

M an R-module $\Rightarrow$ $\widetilde{M}$ an $\mathcal{O}_{\operatorname{Spec} R}$-module
⇝ geometric interpretation of module theory!

Def (Quasi-coherent sheaves)
$\mathcal{F}$ sheaf of $\mathcal{O}_X$-modules on scheme X is quasi-coherent if $\exists$ affine open cover $\{U_i = \operatorname{Spec} R_i : i \in I\}$ of X with

$$\mathcal{F}|_{U_i} \cong \widetilde{M_i} \quad \text{for some } R_i\text{-module } M_i.$$

Rmks
(a) $\mathcal{F}$ q.coh. on X, $U = \operatorname{Spec} R \subseteq X \Rightarrow \mathcal{F}|_U \cong \widetilde{M}$ for $M = \mathcal{F}(U)$
[Hartshorne, Pro II.5.4]
(b) $\mathcal{F}$ is a coherent sheaf if M_i is fin. generated R_i-module above
⇝ don't need this below.

Exq (a) $\mathcal{F} = \mathcal{O}_X$ is q.coherent ($M_i = R_i \Rightarrow \mathcal{O}_{\operatorname{Spec} R_i} \cong \widetilde{R_i}$).
(b) $\mathcal{O}_{\mathbb{P}^n}(d)$ is q.coherent on $\mathbb{P}^n$ (isom. to $\mathcal{O}_{U_i}$ on $U_i \subseteq \mathbb{P}^n$).

Next everything modules do, q.coh. sheaves also do!

Lem $X = \operatorname{Spec} R$ affine scheme
(a) For R-modules M, N we have a bijection

$$\{\text{morph. of } \mathcal{O}_X\text{-modules } \widetilde{M} \to \widetilde{N}\} \xleftrightarrow{1:1} \{R\text{-module homs. } M \to N\}$$

(b) $0 \to M_1 \to M_2 \to M_3 \to 0$ exact $\Leftrightarrow$ $0 \to \widetilde{M_1} \to \widetilde{M_2} \to \widetilde{M_3} \to 0$ exact.

(c) $\widetilde{M} \oplus \widetilde{N} = \widetilde{M \oplus N}$, $\widetilde{M} \otimes \widetilde{N} = \widetilde{M \otimes N}$, for R-modules M, N
$\widetilde{M}^\vee = \widetilde{(M^\vee)}$, for M a fin. generated R-module over a Noetherian ring R.

Consequence

Kernels, images, quotients, direct sums, and tensor products of q. coherent sheaves are q. coherent.

Pf (a) $\tilde{M} \xrightarrow{\tilde{f}} \tilde{N} \xrightarrow[\text{sections}]{\text{global}} M \xrightarrow{f} N$ homom. of $R = \mathcal{O}_X(X)$-modules

Conversely: $M \xrightarrow{f} N \rightsquigarrow M_p \xrightarrow{f_p} N_p \ \forall p \in X \rightsquigarrow \tilde{M} \xrightarrow{\tilde{f}} \tilde{N}$
Constructions are inverse to each other.

(b) $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact seq. of R-modules
$\Longleftrightarrow$ $0 \to (M_1)_p \to (M_2)_p \to (M_3)_p \to 0$ is exact seq. of R_p-modules $\forall p \trianglelefteq R$ prime

[Gathmann, Pr. 6.27]

Stalks of $\tilde{M}_i$ at p

$\rightsquigarrow$ [Lem 13.21] exactness of $0 \to \tilde{M}_1 \to \tilde{M}_2 \to \tilde{M}_3 \to 0$
can be checked at stalks.

(c) [Comm. algebra: $\oplus$, $\otimes$ commutes w/ localization
$(\ )^\vee$ " " " for f.g. mod. / R Noeth.

To prove e.g. $\tilde{M} \oplus \tilde{N} \cong \widetilde{M \oplus N}$:

$U \subseteq X$ open, $(\varphi = (\varphi_p)_{p\in U}, \psi = (\psi_p)_{p \in U}) \in (\tilde{M} \oplus \tilde{N})(U)$

$\varphi_p \in M_p$ $\quad$ $\psi_p \in N_p$

$$\Rightarrow \big(\underbrace{(\varphi_p \oplus \psi_p)}_{\in M_p \oplus N_p \cong (M \oplus N)_p}{}_{p \in U} \big) \in (\widetilde{M \oplus N})(U)$$

$\Rightarrow \tilde{M} \oplus \tilde{N} \longrightarrow \widetilde{M \oplus N}$ morphism of $\mathcal{O}_X$-modules, isom. of stalks at p.
$\Rightarrow$ isom. of $\mathcal{O}_X$-mod.

Then e.g. $\mathcal{F}, \mathcal{G}$ q.coh., $\mathcal{F} \xrightarrow{f} \mathcal{G}$ hom. of $\mathcal{O}_X$-modules
$\Rightarrow U = \operatorname{Spec} R \subseteq X$ we have $\mathcal{F}|_U = \tilde{M}$, $\mathcal{G}|_U = \tilde{N}$
then $\mathcal{F}|_U \xrightarrow{f|_U} \mathcal{G}|_U \Rightarrow M \xrightarrow{F} N$ an R-module hom.

$0 \to \ker(F) \to M \to N \to 0 \rightsquigarrow 0 \to \underbrace{\widetilde{\ker(F)}}_{\cong \ker(f)|_U} \to \tilde{M} \to \tilde{N} \to 0$ $\Rightarrow$ $\ker(f)$ is q.coh. □

The ideal sequence

Have seen $\oplus$, $\otimes$, Ker, Im, ... of q.coh. sheaves is q.coh.

$\rightsquigarrow$ What about pushforwards?

↘ No in general, but hard to construct, not relevant.

Yes for closed subschemes.

Lem (Ideal sequence)

$i: Y \to X$ closed subscheme.

(a) $\mathcal{F}$ q.coh. sheaf on Y $\Rightarrow$ $i_*\mathcal{F}$ q.coh. on X.

(b) There is an exact sequence

$$0 \to \underbrace{\mathcal{J}_{Y/X} \longrightarrow \mathcal{O}_X}_{\text{define } \mathcal{J}_{Y/X} \text{ as kernel.}} \xrightarrow[\uparrow \text{pull-back}]{} i_*\mathcal{O}_Y \longrightarrow 0$$

of q.coh. sheaves on X.

$\mathcal{J}_{Y/X}$: ideal sheaf of Y in X. think: funct. on $U \subseteq X$ that restr. to zero on $U \cap Y$.

Pf (a) First assume $X = \operatorname{Spec} R$ is affine.

$\rightsquigarrow$ $Y = \operatorname{Spec}(R/I) \xhookrightarrow{i} \operatorname{Spec}(R)$ (from $R \to R/I$).

$\mathcal{F}$ q.coh. on Y $\Rightarrow$ $\mathcal{F} = \widetilde{M}$ for an (R/I) module M.

Via $R \to R/I$: can see M as R-module (write M^R)

$\rightsquigarrow$ get $\widetilde{M^R}$ on $\operatorname{Spec} R$.

Claim $i_*\mathcal{F} = \widetilde{M^R}$

Lem $p \in \operatorname{Spec} R \Rightarrow M^R_p = \begin{cases} M_{p \cdot R/I} & \text{, if } p \in Y \subseteq \operatorname{Spec} R \Leftrightarrow p \supseteq I \\ 0 & \text{, otherwise} \end{cases}$

$\Rightarrow$ on $U \subseteq X$ open get map $(i_*\mathcal{F})(U) \longrightarrow \widetilde{M^R}(U)$

$(\varphi_p \in M_{p \cdot R/I})_{p \in i^{-1}(U)} \longmapsto (\varphi^R_p \in M^R_p)_{p \in U}$

$\uparrow = \varphi_p$ if $p \in U \cap Y$, 0 otherwise

$\rightsquigarrow$ isomorphism.

X general $\rightsquigarrow$ cover X by $\operatorname{Spec} R_i$ & apply first part.

(b) $\mathcal{J}_{Y/X}$ <u>defined</u> as kernel $\rightsquigarrow$ only check $\mathcal{O}_X \to i_* \mathcal{O}_Y$ surject.

[Lem 13.21] check on affine open cover $\rightsquigarrow$ assume $X = \operatorname{Spec} R$

$\Rightarrow$ $\underbrace{\mathcal{O}_X}_{\tilde{R}} \longrightarrow i_* \underbrace{\mathcal{O}_Y}_{\widetilde{R/I}}$ given by $\tilde{R} \longrightarrow \widetilde{R/I}$ ↖ see R/I as R-module

$R \to R/I$ surjective $\rightsquigarrow$ $0 \to I \to R \to R/I \to 0$ exact
$\overset{[\text{Lem } 14.7(a)]}{\rightsquigarrow}$ $0 \to \tilde{I} \to \underbrace{\tilde{R} \to \widetilde{R/I}}_{\text{surjective.}} \to 0$ exact

□

<u>Note</u> Proof shows that for $Y = \operatorname{Spec} R/I \xrightarrow{i} X = \operatorname{Spec} R$:
$\mathcal{J}_{Y/X} = \tilde{I}$.

<u>Exa</u> (Skyscraper sequence revisited)
$P = (1:0) \in X = \mathbb{P}^1$, $i : \{P\} \longrightarrow \mathbb{P}^1$ inclusion, $\mathbb{P}^1 = U_0 \cup U_1$ as usual
$\rightsquigarrow$ $i_* \mathcal{O}_{\{P\}} = K_P$ skyscraper sheaf

$$\rightsquigarrow \quad 0 \longrightarrow \underbrace{\mathcal{J}_{\{P\}/X}}_{\doteq\, \mathcal{O}(-1)} \longrightarrow \mathcal{O}_X \longrightarrow \underbrace{i_* \mathcal{O}_Y}_{=\, K_P} \longrightarrow 0 \quad (*) \text{ skyscraper seq. (from before)}$$

<u>Consequences</u>
(a) $\mathcal{J}_{\{P\}/X} \cong \mathcal{O}(-1)$
(b) K_P is quasi-coherent: $\begin{cases} \widetilde{K[x_1]/\langle x_1 \rangle} & \text{on } U_0 \\ \widetilde{\{0\}} & \text{on } U_1 \end{cases}$
(c) Reprove exactness of $(*)$ using q.coh. sheaves:
$\rightsquigarrow$ can check on U_0, U_1
$\rightsquigarrow$ U_0 : corr. to $0 \to K[x_1] \xrightarrow{\cdot x_1} K[x_1] \longrightarrow K[x_1]/\langle x_1 \rangle \longrightarrow 0$
U_1 : corr. to $0 \to K[x_1] \xrightarrow{\text{id}} K[x_0] \longrightarrow 0 \longrightarrow 0$
aff. coord. on U_0, U_1

homogeneous coord. : $0 \to \frac{1}{x_0} K[\frac{x_1}{x_0}] \xrightarrow{\cdot x_1} K[\frac{x_1}{x_0}] \longrightarrow K[\frac{x_1}{x_0}]/\langle \frac{x_1}{x_0} \rangle \longrightarrow 0$
$0 \to \frac{1}{x_1} K[\frac{x_0}{x_1}] \xrightarrow{\cdot x_1} K[\frac{x_0}{x_1}] \longrightarrow 0 \longrightarrow 0$

$\rightsquigarrow$ can check exactness on modules of sections on affine cover

Rmk (Ideal sheaves)

Ideal sheaf = $\mathcal{O}_X$-module $\mathcal{J}$ together with $\mathcal{J} \to \mathcal{O}_X$ injective

(a) Lem: $Y \to X$ closed subscheme $\rightsquigarrow$ $\mathcal{J} = \mathcal{J}_{Y/X}$ is ideal sheaf

Conversely $\mathcal{J}$ q.coherent ideal sheaf $\overset{\text{determ.}}{\rightsquigarrow}$ $Y = V(\mathcal{J}) \to X$ closed subscheme

Indeed: $\{\operatorname{Spec} R_i : i \in I\}$ affine cover $\rightsquigarrow$ $\mathcal{J}|_{\operatorname{Spec} R_i} = \widetilde{\mathcal{J}_i}$ for $\mathcal{J}_i \trianglelefteq R_i$

$\rightsquigarrow$ $\operatorname{Spec} R_i/\mathcal{J}_i \to \operatorname{Spec} R$ closed subschemes $\overset{\text{glue}}{\rightsquigarrow}$ $Y = V(\mathcal{J}) \to X$.

(b) $\mathcal{J}$ q.coh. sheaf of ideals on variety X $\rightsquigarrow$ $\mathrm{Bl}_{\mathcal{J}} X$ blow-up

Construction: $U \subseteq X$ affine open $\rightsquigarrow$ $\mathcal{J}|_U = \widetilde{I}$, $I \trianglelefteq A(U)$ ideal

$\rightsquigarrow$ $\mathrm{Bl}_I U$ defined before $\overset{\text{glue}}{\rightsquigarrow}$ $\mathrm{Bl}_{\mathcal{J}} X$.

Exercise $X \subseteq \mathbb{P}^n$ irred. hypersurf. of degree d

Show: $\mathcal{J}_{X/\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-d)$.

$\boxed{\text{From now on, all sheaves on a scheme are assumed to be quasi-coherent.}}$

Pull-back of sheaves

<u>Have seen:</u> $f: X \to Y$ morphism of schemes, $\mathcal{F}$ sheaf on X
$\Rightarrow f_*\mathcal{F}$ sheaf on Y ($\mathcal{F}$ q.coh, $f: X \to Y$ closed subscheme $\Rightarrow f_*\mathcal{F}$ q.coh.)

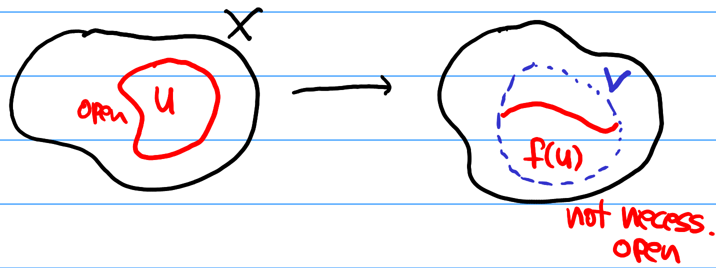
Converse operation: $\mathcal{F}$ (q.coh) sheaf on $Y \rightsquigarrow f^*\mathcal{F}$ sheaf on X?

<u>Construction</u> $f: X \to Y$ morphism of schemes, $\mathcal{F}$ sheaf on Y
(a) Assume $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$ affine $\rightsquigarrow f$ from $S \to R$
$\mathcal{F}$ q.coh. $\Rightarrow \mathcal{F} = \widetilde{M}$ for S-module M

$\rightsquigarrow M$ and R are S-modules $\rightsquigarrow M \otimes_S R$ is R-module

$$f^*\mathcal{F} := \widetilde{M \otimes_S R} \quad \text{on } X \qquad (\text{pull-back of } \mathcal{F} \text{ along } f)$$

(b) X, Y arbitrary



Presheaf on X:

$$\underset{\substack{\cap\!\mathsf{I}\\ X \text{ open}}}{U} \longmapsto \{(V, \varphi) : V \subseteq Y \text{ open w/ } f(U) \subseteq V,\ \varphi \in \mathcal{F}(V)\}/\sim \qquad (\star)$$

where $(V,\varphi) \sim (V', \varphi')$ if $\exists\, W$ open, $f(U) \subseteq W \subseteq V \cap V'$ with $\varphi|_W = \varphi'|_W$

$\rightsquigarrow f^{-1}\mathcal{F}$ sheaf on X = sheafification of $(\star)$

$\rightsquigarrow (f^{-1}\mathcal{F})_p = \mathcal{F}_{f(p)}$ stalks agree

<u>Problem</u> $f^{-1}\mathcal{F}$ not an $\mathcal{O}_X$-module (but $f^{-1}\mathcal{O}_Y$-module!)

$f^{-1}\mathcal{O}_Y$: sheaf of rings on X, $f^{-1}\mathcal{O}_Y \xrightarrow{f^*} \mathcal{O}_X$ hom. of sheaves of rings

$\rightsquigarrow f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$

Observations

- $f^*\mathcal{F}$ is $\mathcal{O}_X$-module:
  φ sect. of $\mathcal{O}_X \rightsquigarrow f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \xrightarrow{id \otimes (\text{mult } \varphi)} f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$

- X, Y affine $\rightsquigarrow$ recover construction in (a)

$X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$ affine, $\mathcal{F} = \widetilde{M}$ for S-module M

For $p \in X$ compute stalks:

$$(f^*\mathcal{F})_p \cong (f^{-1}\mathcal{F})_p \otimes_{(f^{-1}\mathcal{O}_Y)_p} \mathcal{O}_{X,p} \cong \mathcal{F}_{f(p)} \otimes_{\mathcal{O}_{Y,f(p)}} \mathcal{O}_{X,p}$$

$$\overset{[14.2(a)]}{\cong} M_{f(p)} \otimes_{S_{f(p)}} R_p \overset{\text{Lem}}{\cong} (M \otimes_S R)_p$$

$\rightsquigarrow$ Construct morphism of sheaves

$$f^*\mathcal{F} \xrightarrow{\Psi} \widetilde{M \otimes_S R}$$

$$\varphi \in (f^*\mathcal{F})(U) \longmapsto (\varphi_p)_{p \in U} \in \widetilde{M \otimes_S R}(U)$$

use isom. above

Ψ morphism of $\mathcal{O}_X$-modules, isom. on stalks as

$$(f^*\mathcal{F})_p \cong (M \otimes_S R)_p \cong (\widetilde{M \otimes_S R})_p$$

<u>Idea</u> For local computations we use (a)

$\rightsquigarrow$ (b) just needed to see that $f^*\mathcal{F}$ gives well-def. $\mathcal{O}_X$-mod.

Properties of pullback sheaves

$f: X \to Y$ morphism of schemes, $\mathcal{F}$ sheaf on Y

$\rightsquigarrow$ $f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ pull-back sheaf on X

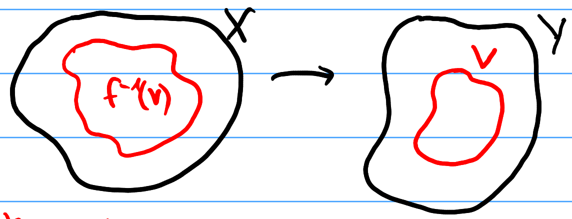
$X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, $\mathcal{F} = \widetilde{M}$ $\rightsquigarrow$ $f^*\mathcal{F} = \widetilde{M \otimes_S R}$

We collect some properties of $f^*\mathcal{F}$:

(a) $f^*\mathcal{O}_Y = f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{O}_X$

$A \otimes_A B \cong B$

(b) Pullbacks of sections



$V \subseteq Y$ open, $\varphi \in \mathcal{F}(V)$

$\rightsquigarrow$ get $f^*\varphi \in (f^*\mathcal{F})(f^{-1}(V))$

$f^*\varphi \in f^*\mathcal{F}(f^{-1}(V)) \leftsquigarrow \varphi \in \mathcal{F}(V)$

Indeed: $f^{-1}\mathcal{F}$ was sheafification of

$$(f^{-1}\mathcal{F}^{PS})(U) = \{(V,\varphi) : V \subseteq Y \text{ open w/ } f(U) \subseteq V,\ \varphi \in \mathcal{F}(V)\}/\sim \qquad (*)$$

$U = f^{-1}(V) \rightsquigarrow (V,\varphi)$ is sect. of $f^{-1}\mathcal{F}^{PS}$ on U

$\rightsquigarrow$ via $(f^{-1}\mathcal{F}^{PS} \to f^{-1}\mathcal{F})$: $(V,\varphi) \in (f^{-1}\mathcal{F})(U)$

$$\Rightarrow f^*\varphi := \underbrace{(V,\varphi)}_{\in (f^{-1}\mathcal{F})(U)} \otimes \underbrace{1}_{\in \mathcal{O}_X(U)} \in (f^*\mathcal{F})(U)$$

Satisfies nice properties:

e.g. $\psi \in \mathcal{O}_Y(V)$ $\rightsquigarrow$ $f^*(\psi \cdot \varphi) = (f^*\psi) \cdot (f^*\varphi)$

mult. in $\mathcal{F}(V)$ mult. in $(f^*\mathcal{F})(f^{-1}(V))$

Pf $(V, \psi\cdot\varphi) \otimes 1 = (V,\varphi) \otimes f^*\psi = (f^*\psi) \cdot (V,\varphi) \otimes 1$

$\psi \in (f^{-1}\mathcal{O}_Y)(f^{-1}(V))$ tensor prod. over $f^{-1}\mathcal{O}_Y$ def. of $\mathcal{O}_X$-module struct. on $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$

(c) Functoriality: $X \xrightarrow{f} Y \xrightarrow{g} Z$, $\mathcal{F}$ an $\mathcal{O}_Z$-module

$\rightsquigarrow$ $f^* g^* \mathcal{F} \cong (g \circ f)^* \mathcal{F}$ as $\mathcal{O}_X$-modules

Pf (affine case) $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, $Z = \operatorname{Spec} T$, $\mathcal{F} = \widetilde{M}$
M T-mod.

$\rightsquigarrow f^* g^* \mathcal{F} = f^* \widetilde{M \otimes_T S} = \widetilde{(M \otimes_T S) \otimes_S R}$

$$\overset{\text{Lem}}{=\!=} \widetilde{M \otimes_T R} = (g \circ f)^* \mathcal{F}$$

(d) Pull-backs by open embeddings

$i_U : U \longrightarrow X$ open embedding $\Rightarrow$ $i_U^{-1} \mathcal{F} \cong \mathcal{F}|_U$

$\Rightarrow i_U^* \mathcal{F} = \mathcal{F}|_U \otimes_{\underbrace{\mathcal{O}_X|_U}_{=\mathcal{O}_U}} \mathcal{O}_U \cong \mathcal{F}|_U$

(e) Fibers at points

$\mathcal{F}$ sheaf on X, $p \in X$ closed point, $i_p : \{p\} \longrightarrow X$ inclus.

$\rightsquigarrow$ $i_p^* \mathcal{F}$ sheaf on $\{p\}$ $\Rightarrow$ $\mathcal{F}|_p := (i_p^* \mathcal{F})(\{p\})$
fiber of $\mathcal{F}$ at p

For factorization
$$\begin{tikzcd} \{p\} \arrow[r, "i_{p,U}"] \arrow[rr, bend right, "i_p"'] & U = \operatorname{Spec} R \arrow[r, "i_U"] & X \end{tikzcd},$$

$\rightsquigarrow i_p^* \mathcal{F} \overset{(c)}{\cong} i_{p,U}^* i_U^* \mathcal{F} \overset{(d)}{\cong} i_{p,U}^* \mathcal{F}|_U = i_{p,U}^* \widetilde{M} = \widetilde{M \otimes_R R/\mathfrak{p}}$

M an R-module   max. ideal assoc. to $p \in U$

$\Rightarrow \mathcal{F}|_p = M/\mathfrak{p}M$ as vector space over $K(p) = R/\mathfrak{p}$

$\varphi \in \mathcal{F}(V)$ for $p \in V \overset{\text{open}}{\subseteq} X \rightsquigarrow \varphi_p \in M_\mathfrak{p} \rightsquigarrow \varphi(p) \in M_\mathfrak{p}/\mathfrak{p} M_\mathfrak{p} \cong \mathcal{F}|_p$

value at p

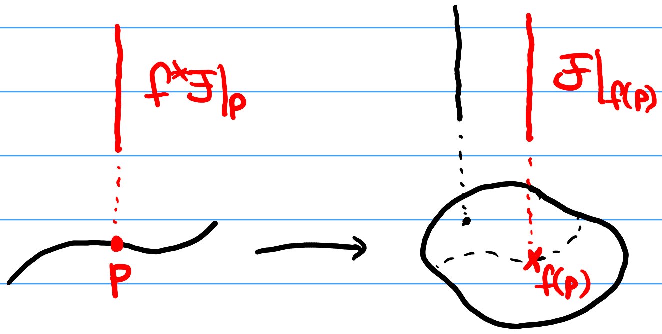
varieties

<u>Interpretation</u> $f: X \longrightarrow Y$, $\mathcal{F}$ q.coh. $\mathcal{O}_X$-mod. on Y

$\rightsquigarrow$ What is fiber $f^*\mathcal{F}|_p$ at $p \in X$ closed point?

Have diagram

$$\begin{tikzcd} \{p\} \arrow[r, "i_p"] \arrow[rr, bend right, "i_{f(p)}"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$



$$\Rightarrow f^*\mathcal{F}|_p = i_p^* f^* \mathcal{F} = i_{f(p)}^* \mathcal{F}$$

$$= \mathcal{F}|_{f(p)}$$

Next: useful class of q.coh. sheaves $\mathcal{F}$ where all fibers $\mathcal{F}|_p$ are isomorphic & fit together nicely.

Locally free sheaves

Easiest class of R-modules: free modules $M = R^n$

Q.coh. sheaves $\mathcal{F}$ on X $\rightsquigarrow$ locally $\tilde{M}$ on $\operatorname{Spec} R$

Def (Locally free sheaves)

$\mathcal{F}$ q.coh. $\mathcal{O}_X$-module is called <u>locally free</u> if

$\exists$ affine cover $\{U_i = \operatorname{Spec} R_i\}$ of X s.t. $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ for M_i free R_i module of fin. rank.

Rank the same for all U_i $\rightsquigarrow$ <u>rank</u> $\operatorname{rk}(\mathcal{F})$ of $\mathcal{F}$.

Rmks

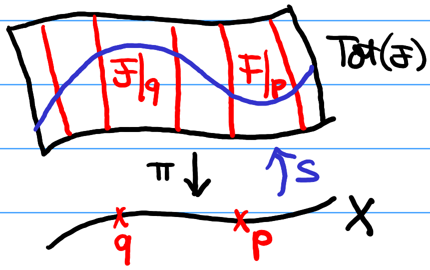
(a) $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r_i}$ but not necessarily $\mathcal{F} \cong \mathcal{O}_X^{\oplus r}$

Exa $\mathcal{O}_{\mathbb{P}^n}(d)$ on $X = \mathbb{P}^n$ $\rightsquigarrow$ have seen: $\mathcal{O}_{\mathbb{P}^n}(d)|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus 1}$

$\rightsquigarrow$ $\mathcal{O}_{\mathbb{P}^n}(d)$ is locally free of rank 1, but $\underbrace{\mathcal{O}_{\mathbb{P}^n}(d) \not\cong \mathcal{O}_{\mathbb{P}^n} \text{ for } d \neq 0}_{\text{e.g. look at global sections}}$

(b) $\mathcal{F}$ locally free of rank r $\rightsquigarrow$ $\mathcal{F}|_p = r$-dim. vect.space $/ k(p)$

$\rightsquigarrow$ $\exists$ scheme $\underset{\text{total space of } \mathcal{F}}{\operatorname{Tot}(\mathcal{F})} \xrightarrow{\pi} X$



such that

$$\mathcal{F}(U) \cong \{ s : U \to \operatorname{Tot}(\mathcal{F}) : \pi \circ s = \mathrm{id}_U \}$$

<u>Idea</u> on U_i with $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$ $\rightsquigarrow$ $\operatorname{Tot}(\mathcal{O}_{U_i}^{\oplus r}) = U_i \times \mathbb{A}^r \longrightarrow U_i$

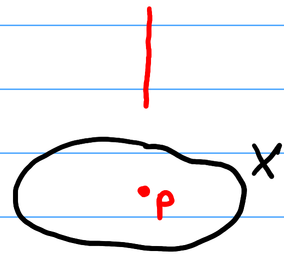
$\rightsquigarrow$ glue those together to $\operatorname{Tot}(\mathcal{F})$ $\rightsquigarrow$ <u>vector bundle</u>

$r = 1$: $\mathcal{F}$ <u>invertible sheaf</u>, $\operatorname{Tot}(\mathcal{F})$: <u>line bundle</u>

Exa $\mathcal{O}_X^{\oplus r}$ (trivial) locally free sheaf of rank r

$\rightsquigarrow \mathrm{Tot}(\mathcal{O}_X^{\oplus r}) = X \times \mathbb{A}^r \xrightarrow{\pi} X$ (trivial) vector bundle

Non-Exa $\mathcal{F} = K_p$



$$\rightsquigarrow \mathcal{F}|_q = \begin{cases} K & , q = p \\ 0 & , q \neq p \end{cases} \rightsquigarrow \mathcal{F} \text{ not loc. free}$$

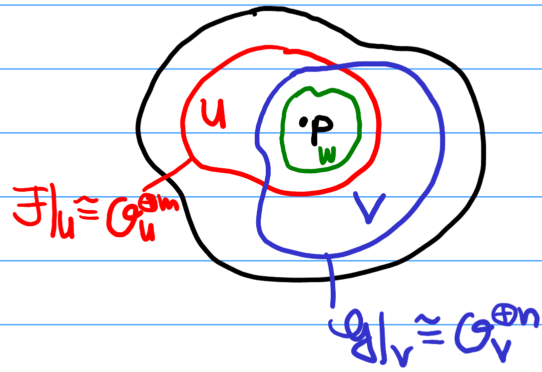
(unless $\{p\}$ is open...)

Pro $\mathcal{F}, \mathcal{G}$ locally free sheaves of ranks m, n, then

- $\mathcal{F} \oplus \mathcal{G}$ loc. free of rank $m+n$
- $\mathcal{F} \otimes \mathcal{G}$ loc. free of rank $m \cdot n$
- $\mathcal{F}^\vee$ loc. free of rank m
- $f^*\mathcal{F}$ loc. free of rank m for $f: Y \to X$ morphism.

Pf $\forall p \in X$ $\exists$ open nbhd. U, V with

$\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus m}$, $\mathcal{G}|_V \cong \mathcal{O}_V^{\oplus n}$



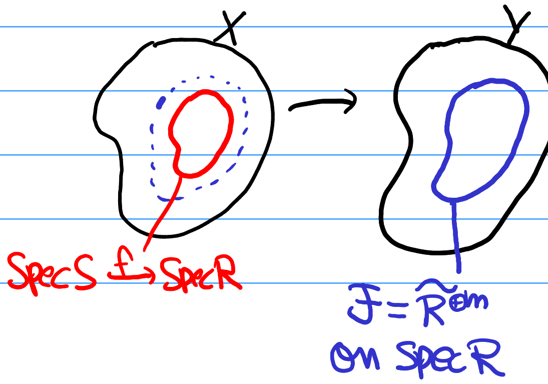
On $p \in W \subseteq U \cap V$, $W = \mathrm{Spec}\, R$ affine

$$\mathcal{F} \oplus \mathcal{G}|_W = \widetilde{R^{\oplus m} \oplus R^{\oplus n}} \cong \mathcal{O}_W^{\oplus (m+n)}$$

$$\mathcal{F} \otimes \mathcal{G}|_W = \widetilde{R^{\oplus m} \otimes R^{\oplus n}} \cong \mathcal{O}_W^{\oplus (m \cdot n)}$$

$$\mathcal{F}^\vee|_W = \widetilde{\mathrm{Hom}_R(R^{\oplus m}, R)} \cong \widetilde{R^m} \cong \mathcal{O}_W^{\oplus m}$$

Finally (in situation on right):



$$f^*\mathcal{F}|_{\mathrm{Spec}\, S} = \widetilde{R^{\oplus m} \otimes_R S} = \widetilde{S^{\oplus m}} = \mathcal{O}_{\mathrm{Spec}\, S}^{\oplus m}.$$

$\square$

Application: maps to projective space

Recall

(a) $\mathcal{O}_{\mathbb{P}^n}(d)$ invertible sheaf (line bundle) on $\mathbb{P}^n = \mathbb{P}^n_K$

$\rightsquigarrow \mathcal{O}_{\mathbb{P}^n}(d)(\mathbb{P}^n) = K[x_0,\dots,x_n]_d = S(\mathbb{P}^n)_d$ global sections

$\mathcal{O}_{\mathbb{P}^n}(1)(\mathbb{P}^n) = \mathrm{Lin}_K(x_0,\dots,x_n)$

(b) $f_0,\dots,f_n \in S(\mathbb{P}^m)_d$ homogeneous of same degree

$\rightsquigarrow (f_0:\dots:f_n) : \mathbb{P}^m \setminus V_p(f_0,\dots,f_n) \longrightarrow \mathbb{P}^n$ morphism

can interpret as sections of $\mathcal{L} = \mathcal{O}_{\mathbb{P}^m}(d)$

Thm Given a variety X over K, there is a bijection

$$\left\{ \begin{matrix} \text{morphisms} \\ f: X \longrightarrow \mathbb{P}^n_K \end{matrix} \right\} \xrightarrow[\Psi]{\sim} \left\{ (\mathcal{L}, s_0,\dots,s_n) \;\middle|\; \begin{matrix} \mathcal{L}/X \text{ invertible sheaf,} \\ s_0,\dots,s_n \in \mathcal{L}(X) \text{ sect.} \\ \text{s.t. } \forall p \in X\ \exists i : s_i(p) \neq 0 \end{matrix} \right\} \Big/ \text{isom.}$$

$$f \longmapsto (\mathcal{L} = f^*\mathcal{O}(1),\ s_0 = f^* x_0, \dots, s_n = f^* x_n)$$

Here $(\mathcal{L}, s_0,\dots,s_n) \cong (\mathcal{L}', s_0',\dots,s_n')$ if $\exists\ \mathcal{L} \cong \mathcal{L}'$ sending s_i to s_i'.

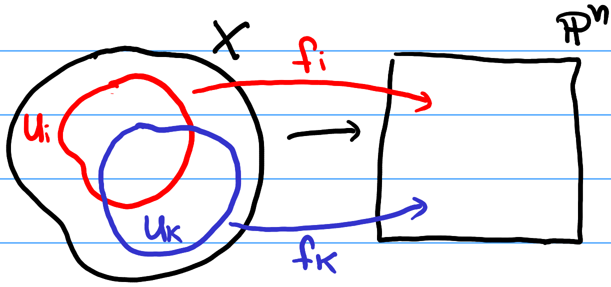
Lem Y scheme, $\Psi : \mathcal{O}_Y \xrightarrow{\sim} \mathcal{O}_Y$ isom.
$\Rightarrow s = \Psi(1) \in \mathcal{O}_Y(Y)^\times$ and $\Psi(\varphi) = s \cdot \varphi \quad \forall \varphi \in \mathcal{O}_Y(Y)$.
Pf $\Psi(\varphi) = \Psi(\varphi \cdot 1) = \varphi\, \Psi(1) = \varphi \cdot s$ since Ψ = hom. of $\mathcal{O}_Y$-mod.
$t = \Psi^{-1}(1) \rightsquigarrow 1 = \Psi(\Psi^{-1}(1)) = s \cdot t \Rightarrow s \in \mathcal{O}_Y(Y)^\times$ invertible. $\diamond$

Pf of Thm $\boxed{\Psi \text{ well-def.}}$ $\mathcal{O}_{\mathbb{P}^n}(1)$ loc. free of rk 1 $\Rightarrow \mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ l.f. rank 1.
$x_i \in \mathcal{O}_{\mathbb{P}^n}(1)(\mathbb{P}^n) \quad \Rightarrow s_i = f^* x_i \in \mathcal{L}(X)$.

$s_i(p) = 0 \in \mathcal{L}|_p \cong \mathcal{O}_{\mathbb{P}^n}(1)|_{f(p)} \Leftrightarrow \underbrace{x_i|_{f(p)} = 0}_{\overset{\text{check}}{\Leftrightarrow} f(p) \in V(x_i)}$ but $V(x_0,\dots,x_n) = \emptyset$
$\rightsquigarrow$ not all $s_i(p)$ can be 0.

$\boxed{\text{Inverse construction}}$ $(\mathcal{L}, s_0, \dots, s_n)$ as above

$\mathcal{L}$ loc free rk 1
$\Rightarrow \exists$ cover $\{U_i : i \in I\}$ of X:
$\mathcal{L}|_{U_i} \underset{\varphi_i}{\cong} \mathcal{O}_{U_i} \rightsquigarrow s_j \underset{\varphi_i}{\cong} f_{j,U_i} \in \mathcal{O}_{U_i}(U_i)$
regular funct.



$\Rightarrow f_i = (f_{0,U_i} : \dots : f_{n,U_n}) : U_i \longrightarrow \mathbb{P}^n$ well-def. morph. $(\forall p \exists j : f_j(p) \neq 0)$

On $U_i \cap U_k$:
$$\underbrace{\mathcal{O}_{U_i \cap U_k} \underset{\varphi_i}{\cong} \mathcal{L}|_{U_i \cap U_k} \underset{\varphi_k}{\cong} \mathcal{O}_{U_i \cap U_k}}_{\varphi_{ik}, \ \sim}$$

<u>Lem</u> $\Rightarrow \varphi_{ik}$ given by multiplic. with $s_{ik} \in \mathcal{O}_{U_i \cap U_k}(U_i \cap U_k)^\times$

$$\Rightarrow f_i|_{U_i \cap U_k} = (f_{0,U_i} : \dots : f_{n,U_i}) = (s_{ik} f_{0,U_k} : \dots : s_{ik} f_{n,U_k})$$
$$= (f_{0,U_k} : \dots : f_{0,U_k}) = f_k|_{U_i \cap U_k} \rightsquigarrow (f_i)_{i \in I} \text{ glue to } f : X \to \mathbb{P}^n$$

$\mathbb{P}^n$ invar. under scaling by $K^\times$

Similar: $(\mathcal{L}', s_0', \dots, s_n')$ isom. $\rightsquigarrow f'_{j,U_i}$ differ from f_{j,U_i} common factor s_i.

<u>Exercise</u> The two constructions above are inverse.
[see Vakil, Sect. 15.2] $\square$

$\rightsquigarrow$ Can use invertible sheaves / line bundles to understand classical questions on varieties (maps to $\mathbb{P}^n$)

<u>Thm</u> Every line bundle $\mathcal{L}$ on $\mathbb{P}^n$ is of the form $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^n}(d)$ for some d.

<u>Exercise</u> Show $\mathrm{Aut}(\mathbb{P}^n_K) \cong PGL_K(n+1)$.